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ON FERMION LOOPS OF TWO VERTICES

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ABSTRACT. The imaginary part of the retarded matrix element for a closed loop two vertices has been deduced by perturbation theory. It has been used to evaluate the photon and meson vacuum polarisation effects by assuming dispersion relations. Some difficulties regarding the mesic vacuum polarisation have been shown to be removed by considering the vertex correction.

The formulae have been applied to deduce the decay rates of some fundamental particles and the results obtained are in good agreement with experiments.

INTRODUCTION

Much interest is attached to explain the decay of fundamental particles by weak universal fermi interaction and the known strong/medium strong interactions. In most of the decay processes, a closed loop has to be inserted in Feynman diagram to bring in the four-fermion vertex, a convenient example being the charged pion decay. Since the decay probability is proportional to the weak coupling constant (10^{-14}n. u.) squared, one would expect that a straightforward perturbation analysis using Dyson's S-Matrix expansion will give us the correct decay rates. However, the closed loops intervening the initial and final states bring in infinite constants. They cannot be renormalised in a formal way i.e. we cannot absorb the infinite constants as unobservable constants associated with mass and coupling constant. Most of the fundamental-particle-decay studies have, therefore, been confined to the ratio of decay rates and the like. And only very recently $\pi^{\pm} \rightarrow \mu^{\pm} + \nu(\bar{\nu})$ has been studied with considerable success by Goldberger and Trieman (1958).

To get rid of the infinite constants we shall calculate the imaginary part of the retarded matrix elements. This part is usually free from infinities and the real part can be evaluated from it by dispersion relations. In fact we shall show that the observable finite matrix elements can be unambiguously obtained from the imaginary part alone. Of course, one has to make a few subtractions, a procedure reminiscent of Pauli-Villar Regularisation Scheme.

MATRIX ELEMENT OF A LOOP

Let us take a loop with two vertices ($g_1\Gamma_1$; $g_2\Gamma_2$); $g_1 g_2$ denoting the coupling strength and Γ_1 and Γ_2 may consist of matrices. Referring to the Feynman diagram in Fig. 1, the contribution to the matrix element can be written as

$$F_{12}(k)\propto\frac{g_1g_2}{i\pi^2}\int d^4p\,\frac{Tr.[\Gamma_1(i\gamma\,.\,p-m)\Gamma_2(i\gamma.p-k-m)]}{(p^2+m^2-i\epsilon)\,(p-k)^2+m^2-i\epsilon)} \qquad \dots (1)$$

where m is the mass of the fermion. Evaluating the trace and combining the denominators we get

$$F_{12}(k)\propto\frac{g_2g_1}{i\pi^2}\int_0^1 dx\int d^4p\,\frac{Ap^2+B_{\mu\nu}k_\mu k_\nu x(1-x)-\frac{1}{2}(C_\mu-D_\mu)imk_\mu+Em^2}{(p^2+k^2x(1-x)+m^2-i\epsilon)^2} \qquad \dots (2)$$

where

$$\begin{aligned} A &= -\tfrac{1}{4}sp(\Gamma_1\gamma_\mu\Gamma_2\gamma_\mu)\\ B_{\mu\nu} &= sp(\Gamma_1\gamma_\mu\Gamma_2\gamma_\nu)\\ C_\mu &= sp(\Gamma_1\gamma_\mu\Gamma_2)\\ D_\mu &= sp(\Gamma_1\Gamma_2\gamma_\mu)\\ E &= sp\,.\,(\Gamma_1\Gamma_2) \end{aligned} \qquad \dots (2a)$$

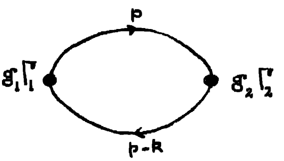

Fig 1.

After partial integration over 'x', $F_{12}(k)$ can be written as (omitting g_1, g_2)

$$F_{12}(k)=\frac{A}{i\pi^2}\int\frac{d^4p}{p^2+m^2-i\epsilon}+\left(\frac{1}{6}B_{\mu\nu}k_\mu k_\nu-\frac{1}{3}Ak^2-\frac{1}{2}im\,k_\mu(C_\mu-D_\mu)\right.$$

$$\left.+(E-A)m^2\right)\frac{1}{i\pi^2}\int\frac{d^4p}{(p^2+m^2-i\epsilon)^2}+\int_0^1 dx\left\{B_{\mu\nu}k_\mu k_\nu\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\right.$$

$$\left.-Ak^2\frac{x^3}{3}-\frac{1}{2}(C_\mu-D_\mu)\,im\,k_\mu x+(E-A)m^2x\right\}\frac{k^2(1-2x)}{k^2x(1-x)+m^2-i\epsilon} \qquad \dots (3)$$

It is easily seen that the last term alone has an imaginary part, since

$$\frac{1}{a-i\epsilon}=\frac{P}{a}+i\pi\,\delta(a)$$

$$\therefore \quad Im\,.\,F_{12}(k^2) = +\pi \int_0^1 dx\, \delta(k^2x(1-x)+m^2)\, k^2(1-2x)\,.$$

(expression in { } of Eqn. 3) ... (4)

Im part of the retarded $F_{12}(k^2)$ can be obtained by multiplying $\epsilon(k_0)$,

since
$$\Delta_F(k) = \frac{1}{k_\mu{}^2+a^2-i\epsilon} = \frac{\rho}{k^2+a^2} + i\pi\,\delta(k^2+a^2)$$

and
$$\Delta_R(k) = \frac{1}{\vec{k}^2-(k_0+i\epsilon)^2+Q^2} = \frac{\rho}{k^2+a^2} + i\pi\epsilon(k_0)\delta(k^2+a)$$

with
$$\begin{aligned} \epsilon(k_0) &= +1\,, & k_0 &> 0 \\ &= 0\,; & k_0 &= 0 \\ &= -1\,; & k_0 &< 0 \end{aligned} \qquad \dots (5)$$

Our task is now to integrate over x with proper care to the delta function.

To this end we write

$$\delta(k^2x(1-x)+m^e) = \frac{1}{4m^2x(1-x)}\,\delta\left(\frac{k^2}{4m^2}+\frac{4}{x(1-x)}\right).$$

and make the transformation $x = \frac{1+u}{2}$. The resulting integration is symmetrical and we get

$$Im\,.\,F_{12}^R(k^2) = 4\pi\epsilon(k_0)\int_0^1 \delta\left(\frac{k^2}{4m^2}+\frac{1}{1-u^2}\right)\frac{k^2}{4m^2}\,\frac{u^2}{1-u^2}\left\{\frac{1}{12}B_{\mu\nu}k_\mu k_\nu\left(1+\frac{1-u^2}{2}\right)\right.$$

$$\left.+\frac{1}{6}Ak^2\left(\frac{1-u^2}{4}-1\right)-\frac{1}{4}(C_\mu-D_\mu)imk_\mu+\frac{1}{2}(E-A)m^2\right\}du \qquad \dots (6)$$

Letting $z = \frac{1}{1-u^2}$ with the help of a theta function

$$\begin{aligned} \theta(x) &= 1; & x &> 0 \\ &= 0, & x &< 0 \end{aligned}$$

$$Im\,.\,F_{12}^R(k^2) = 4\pi\epsilon(k_0)\int_{-\infty}^{+\infty}\theta(z-1)\,\delta\left(\frac{k^2}{4m^2}+z\right)\frac{k^2}{4m^2}\,\frac{dz}{2z}\sqrt{1-\frac{1}{z}}\left\{\frac{1}{12}B_{\mu\nu}k_\mu k_\nu\right.$$

$$\left.\left(1+\frac{1}{2z}\right)+\frac{1}{6}Ak^2\left(\frac{1}{4z}-1\right)-\frac{1}{4}(C_\mu-D_\mu)imk_\mu+\frac{1}{2}(E-A)m^2\right\}$$

$$= -\frac{\pi}{2}\epsilon(k_0)\,4\theta\left(-\frac{k^2}{4m^2}-1\right)\sqrt{1+\frac{4m^2}{k^2}}\left\{\frac{1}{12}B_{\mu\nu}k_\mu k_\nu\left(1-\frac{2m^2}{k^2}\right)\right.$$

$$\left.-\frac{1}{6}Ak^2\left(\frac{m^2}{k^2}+1\right)-\frac{1}{4}imk_\mu(C_\mu-D_\mu)+\frac{1}{2}(E-A)m^2\right\} \qquad \dots (7)$$

This, then, is the divergence-free retrate matrix element contribution and small correspond to the imaginary part of the total matrix element.

PHOTON VACUUM POLARISATION

The vacuum polarisation tensor $\Pi_{\mu\nu}$ of a photon for electron loop is given by

$$\Pi_{\mu\nu} = \frac{ie^2}{(2\pi)^4}\int d^4p\, Tr.\left\{ \gamma_\mu \frac{i\gamma\,.\,\overline{p-k}-m}{(p-k)^2+m^2-i\epsilon}\gamma_\nu \frac{i\gamma\,.\,p-m}{p^2+m^2-i\epsilon}\right\}$$

Thus

$$\Gamma_1 = \gamma_\mu\,; \quad \Gamma_2 = \gamma_\nu$$

$$A = 2g_{\mu\nu}\,; \quad B_{\lambda\sigma} = 4(g_{\lambda\mu}g_{\sigma\nu} - g_{\mu\nu}g_{\lambda\sigma} + g_{\lambda\nu}g_{\mu\sigma})\,; \quad C_\mu = D_\mu = 0\,; \quad E = 4g_{\mu\nu}.$$

So we can write

$$Im.\, \pi^R_{\mu\nu}(k^2) = (k_\mu k_\nu - g_{\mu\nu}k^2) \quad Im\,.\,\pi^R(k^2)$$

$$Im\,.\,\pi^R(k^2) = -\frac{\alpha}{3}\,\theta\left(-\frac{k^2}{4m^2}-1\right)\sqrt{1+\frac{4m^2}{k^2}}\left(1-\frac{2m^2}{k^2}\right)\epsilon(k_0) \qquad \ldots (8)$$

For large k^2 this imaginary part of $\pi^R(k^2)$ approaches a constant value, so the Hilbert transform will be logarithimically divergent, necessitating one subtraction.

$$Re\,\pi_f(k^2) = Re\,.\,[\pi(k^2)-\pi(0)] = -\frac{\alpha}{3\pi}\,k^2 p\int_{4m^2}^{\infty}\frac{\left(1+\frac{4m^2}{x}\right)^{\frac{1}{2}}\left(1-\frac{2m^2}{x}\right)}{x(x-k^2)}dx$$

$$= -\frac{\alpha}{3\pi}\left[\frac{5}{3} - \frac{4m^2}{k^2} - \left(1-\frac{2m^2}{k^2}\right)\sqrt{1+\frac{4m^2}{k^2}}\,\log\frac{\sqrt{1+\frac{4m^2}{k^2}}+1}{\sqrt{1+\frac{4m^2}{k^2}}-1}\right] \ldots (9)$$

which is the correct result.

MESIC VACUUM POLARISATION

Here $\Gamma_1 = \Gamma_2 = \gamma_5$ for a pseudoscalar coupling.

$$A = -4; \qquad B_{\mu\nu} = 4g_{\mu\nu}$$
$$C_\mu = D_\mu = 0; \qquad E = -4$$

Denoting the polarisation by $F(k^2)$ we have

$$Im.\text{ part of } F(k^2) = -\frac{G^2}{4\pi}\,\theta\left(\frac{k^2}{4m^2}-1\sqrt{1+\frac{4m^2}{k^2}}\left(\frac{1}{8}\,k^2\right)\right)$$

This diverges linearly with $k^2\to\infty$.

Hence the real part is quadratically divergent. Two subtractions are needed. The net observable effect is therefore contained in the integral

$$(G^2/4\pi) \cdot \frac{\rho}{\pi} \int_{4m^2}^{\infty} \frac{\sqrt{1-\frac{4m^2}{x}}\ \frac{1}{8}\ x}{(x+\mu^2)^2(x-k^2)} (k^2+\mu^2)^2\, dx. \qquad \ldots \quad (11)$$

Evaluation of this integral will lead directly to the finite part of the meson propagation function.

EFFECT OF VERTEX CORRECTION ON THE CLOSED LOOP

In this section we shall study the correction due to the two vertices of the loop. Imaginary parts as given in equations (9) and (8) represent the effect of real pairs in the intermediate states. It should be obvious, therefore, :the matrix elements of the vertex should be evaluated for real intermediate states; but for all values of k^2 subject to the condition that $k^2 \leqslant -4m^2$. For electrodynamics, this amounts to a negligible correction. But for mesons coupled to necleons this amounts to a large correction. It should be noted that since the charge states of the nucleons are fixed, the correction for charged pion vertex is solely due to emission and absorption of π^0 mesons. The details of evaluation have been given in the Appendix. The result obtained is to replace γ_5 by a new function.

$$\Gamma_5 = \frac{\gamma_5}{1-\lambda\frac{k^2}{4m^2_p}} \qquad \text{where } \lambda = \frac{g^2}{24\pi}\left(\frac{m_p}{\mu}\right)^2 \qquad \ldots \quad (12)$$

$$g^2 = \frac{G^2}{4\pi}.$$

Taking $g^2 = 15$; the coefficient $\lambda \simeq B$. This amounts to a large reduction to the real pair formation at the vertex. Even at threshold, $k^2 = -4m_p^2$; this is as large as 90%.

DECAY OF FUNDAMENTAL PARTICLES

We are now in a position to cal culate the absolute decay rates which involve a close loop of which one vertex representing a strong γ_5 interaction. Let us take first the decay

$$\pi^+ \rightarrow \mu^+ + \nu$$

$$\pi^- \rightarrow \mu^- + \bar{\nu}$$

The Lagrangian density describing these decays is

$$\alpha_{int} = f_A(\bar{\Psi}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5\psi_\mu)(\bar{\Psi}_n i\gamma_\lambda\gamma_5\psi_p)$$

$$+f_V(\bar{\Psi}_\nu(1-\gamma_5)\gamma_\lambda\psi_\mu)(\bar{\Psi}_n\gamma_\lambda\psi_p)+i\sqrt{2}G\bar{\Psi}_n\gamma_5\psi_p\phi_{\pi+}$$

$$+iG\,\bar{\Psi}_n\gamma_5\psi_p\phi_{\pi 0}$$

$$+\text{Hermitian conjugate.} \qquad \dots \quad (13)$$

Applying perturbation we shall take the Fermi coupling once and correct for the strong coupling vertex. First we note the vector coupling gives zero, due to the 'spur', so we have

$$\Gamma_1 = \gamma_5 \; : \; \Gamma_2 = \gamma_\lambda\gamma_5$$

for the charged pion decay.

$$A = 0$$
$$B = 0$$
$$E = 0$$
$$C_\mu = -D_\mu = 4g_{\mu\lambda}.$$

Equation (7) takes the form

$$Im\,.F_{12}(k^2) = 4im_p k_\lambda \pi\; \epsilon(k_0)\; \theta\left(-\frac{k^2}{4m_p{}^2}-1\right)\sqrt{1+\frac{4m_p{}^2}{k^2}}$$

The factor ik_λ is to be contracted $i\gamma_\lambda k_\lambda$ with the free particle (μ, ν) spinors yielding m_μ, the mass of the muon. Thus

$$F(k^2) = 4im_p\,.\,k_\lambda\pi\frac{\rho}{\pi}\int_{4m_p{}^2}^{\infty}\frac{\sqrt{1-\frac{4m_p{}^2}{x}}}{x-k^2}\,dx \qquad \dots\dots \quad (14)$$

which is log-divergent. However, if the correction at the vertex 1 is taken into account as given by (12),

$$F(k^2) = 4im_p k_\lambda\int_{4m_p{}^2}^{\infty}\frac{\sqrt{1-\frac{4m_p{}^2}{X}}}{\left(1+\frac{\lambda}{4m_p{}^2}X\right)(X-k^2)}\,dX \qquad \dots \quad (15)$$

which is convergent We get

$$F(k^2) = im_p k_\lambda \cdot \frac{4m^2/k^2}{\lambda + \frac{4m^2}{k^2}} \cdot \left[(1+\lambda)^{\frac{1}{2}} ln. \left| \frac{(1+\lambda)^{\frac{1}{2}}+1}{(1+\lambda)^{\frac{1}{2}}-1} \right| \right.$$

$$\left. - \left(1 - \frac{4m^2}{k^2} \right)^{\frac{1}{2}} \log \left| \frac{\left(1 - \frac{4m^2}{k^2} \right)^{\frac{1}{2}} + 1}{\left(1 - \frac{4m^2}{k^2} \right)^{\frac{1}{2}} - 1} \right| \right] \quad \ldots \quad (16)$$

We shall frequently use this result in evaluating decay rates. For convenience we write

$$F(-\mu^2) = 8im_p k_\lambda \, G(-\mu^2) \quad \ldots \quad (17)$$

where

$$G(-\mu^2) = \frac{1}{2} \frac{4m^2/\mu^2}{4m^2/\mu^2 - \lambda} \left[(1+\lambda)^{\frac{1}{2}} ln. \left| \frac{(1+\lambda)^{\frac{1}{2}}+1}{(1+\lambda)^{\frac{1}{2}}-1} \right| \right.$$

$$\left. - \left(1 + \frac{4m^2}{\mu^2} \right)^{\frac{1}{2}} ln. \left| \frac{\left(1 + \frac{4m^2}{\mu^2} \right)^{\frac{1}{2}} + 1}{\left(1 + \frac{4m^2}{\mu^2} \right)^{\frac{1}{2}} - 1} \right| \right] \quad \ldots \quad (17a)$$

We can estimate $G(-\mu^2)$ by noting

$$4m^2/\mu^2 >> \lambda > 1.$$

Expanding the logarithims and retaining the first leading term, we obtain

$$G(-\mu^2) = \frac{1}{1+\lambda} \quad \ldots \quad (18)$$

which depends only on the damping at the vertex.

Returning to the calculation of the decay rate we note the matrix element is in understandable notation

$$<\mu^+, \bar{\nu} \,|\, M \,|\, \pi^+> = -\frac{\sqrt{2G}\, g_A}{(2\pi)^{9/2}} \frac{1}{\sqrt{2\omega_\pi}} \sqrt{\frac{m_\mu}{\epsilon_\mu \epsilon_\nu}} \, \bar{u}_\mu(p_\mu)(ir.k_\pi)(1+\gamma_5)v_\nu(p_\nu).$$

$$8m_p \pi^2 \, G(-\mu^2). \quad \ldots \quad (19)$$

with four momentum conservation.

This leads to the following decay rate in a familiar way,

$$\omega = \frac{m_\pi}{2\pi^4} \left(\frac{m_\mu}{m_\pi} \right)^2 \left(\frac{m_p}{m_\pi} \right)^2 \left(1 - \frac{m_\mu^2}{m_\pi^2} \right)^2 \left(\frac{G^2}{4\pi} \right) \cdot (g_A m_\mu^2)^2 [G(-\mu^2)]^2 \tag{20}$$

If we take $G^2/4\pi = 15$ and if we adopt for g_A the value of the Gomow-Teller coupling constant, we then find from the known pion life time that

$$G(-\mu^2) = 0.13. \tag{21}$$

whereas equation (18) gives

$$G(-\mu^2)\,|_{theory} = 0.11. \tag{22}$$

Thus the agreement can be considered extremely satisfactory. We emphasize again the influence of vertex damping as has been pointed out by Goldberger and Triemann (1958); they have, however, made a phase-shift analysis whereas we have just corrected for the vertex.

One would naturally be tempted to apply similar considerations to $K \to \mu + \nu$ decays. Ignoring strangeness considerations, the decay rate is not very different from what one would expect. The value of λ, however, is about 10 times smaller, and the full expression (17a) is to be used. One cannot compare the life time with experiment since there are many channels of decay for the K-meson.

We shall extend our calculations to the decay modes

$$\Delta \to \begin{cases} p+\pi^- & \dots \quad (23a) \\ n+\pi^0 & \dots \quad (23b) \end{cases}$$

The simplest possible graphs leading to pionic Δ-decay are shown in Fig. 2.

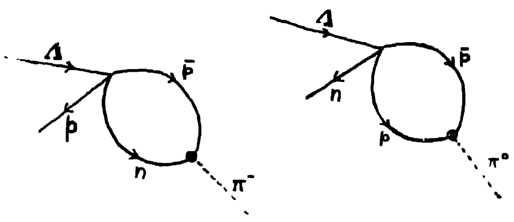

Fig. 2.

The decay rates are

$$\omega(\Delta \to p+\pi^-) = g_A^2 \frac{G^2}{4\pi} \frac{1}{2\pi^4} [G(-\mu^2)]^2 m_p^3 (m_\Delta - m_p)^2 (m_\Delta^2 + m_p^2 - \mu^2) \frac{1}{m_\Delta^3}$$

$$\cdot [(m_\Delta - m_p)^2 - \mu^2]^{\frac{1}{2}} [(m_\Delta + m_p)^2 - \mu^2]^{\frac{1}{2}}$$

$$\frac{\omega(\Delta \to p+\pi^0)}{\omega(\Delta \to p+\pi^-)} = 0.5.$$

which is very near the observed ratio 0.59 ± 0.07. In deducing this we have tacitly assumed that (p, p) are the only intermediate virtual pair for the decay involving the neutral meson.

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APPENDIX

Here we shall deduce the correction to a vertex as shown in Fig. 3, so that all the corrections due to pulling out the vertex in the direction of the boson line are accounted for.

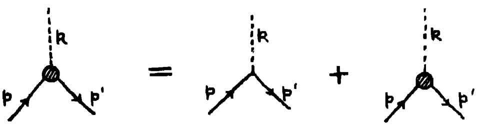

Fig. 3.

The function Γ_5 for such processes as depicted is

$$\Gamma_5 = \gamma_5 + \frac{iG^2}{(2\pi)^4}\int d^4l \,.\, \frac{\gamma_5(ir\,.\,\overline{p-l}-m_p)\Gamma_5(ir\,.\,\overline{p'-l}+m_p)\gamma_5}{(l^2+\mu^2)((p-l)^2+m_p^2)((p'-l)^2+m_p^2)}$$

Since Γ_5 is proportional to odd powers of γ_5 we can anticommute it through the γ-matrices. The denominators can be combined by the well-known Feynman method and we obtain,

$$\Gamma_5 = \gamma_5 + \frac{iG^2}{(2\pi)^4}\, 2\int_0^1 dx \int_0^x dy \int d^4l \,.\, \frac{(ir\,.\,\overline{p-l}+m_p)\,\Gamma_5(ir\,.\,\overline{p'-l}+m_p)}{[(l-px-ky)^2+a^2]^3}$$

where

$$a^2 = m_p^2x^2+\mu^2(1-x-y(x-y))+(p^2+m^2)(x-y)(1-x)+(p'^2+m^2)(1-x)y$$
$$+(k^2+\mu^2)y(x-y)$$

Shifting the origin of '1' integration, we can write,

$$\Gamma_5 = \gamma_5 + \frac{iG^2}{(2\pi)^4}\, 2\int_0^1 dx \int_0^x dy \int d^4l \,.\, \frac{K-(l^2-m_p^3x-\mu^2y(x-y))\Gamma_5}{(l^2+a^2)^3}$$

where,

$$K = -(i\gamma . p' + m_p)\Gamma_5(i\gamma . p+m_p-i\gamma \; pX-i\gamma . kY)$$
$$+(i\gamma . pX+i\gamma . kY) \;\; \Gamma_5(i\gamma . p+m_p)-(p^2+m_p^2)x(x-y)\Gamma_5+(p'^2+m_p^2)xy \; \Gamma_5$$
$$-(k^2\mu^2)\,Y(X-Y)\Gamma_5.$$

One can now renormalize in the conventional way, by writing,

$$\Gamma_5(p', p) = \gamma_5(1+L)+\Gamma_{5f}\,(p', p)$$

where L is determined by the equation

$$\Gamma_{5f}(p, p) = 0 \quad \text{for} \quad i\gamma . p' = i\gamma . p = -m_p.$$

For our case, $\Gamma_{5f}(p', p) = \Gamma_5\, I(k^2)$ with $i\gamma . p' = i\gamma . p = -m_p$, but for arbitrary value of k^2.

Performing the necessary 'I' integration and estimating the major contributions from y and x integrations, we obtain the equation (12) given in the text.

REFERENCE

Goldberger M. L. and Treiman, S. B., 1958 *Phys. Rev.*, **110**, 1178